

Are incomplete and self-confident preference relations better in multicriteria decision making? A simulation-based investigation¹

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Abstract: Incomplete preference relations and self-confident preference relations have been widely used in multicriteria decision-making problems. However, there is no strong evidence, in the current literature, to validate their use in decision-making. This paper reports on the design of two bounded rationality principle based simulation methods, and detailed experimental results, that aim at providing evidence to answer the following two questions: (1) what are the conditions under which incomplete preference relations are better than complete preference relations?; and (2) can self-confident preference relations improve the quality of decisions? The experimental results show that when the decision-maker is of medium rational degree, incomplete preference relations with a degree of incompleteness between 20% and 40% outperform complete preference relations; otherwise, the opposite happens. Furthermore, in most cases the quality of the decision making improves when using self-confident preference relations instead of incomplete preference relations. The paper ends with the presentation of a sensitivity analysis that contributes to the robustness of the experimental conclusions.

Keywords: Decision processes; Incomplete preference relations; Self-confidence; Rational degree; Incomplete degree.

1. Introduction

Preference relations are one of the most widely used preference representation structures in multicriteria decision-making problems [7, 27, 30]. Various types of preference relations have been proposed in the literature, such as additive preference relations (also called fuzzy

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preference relations) [20, 28, 33, 40, 48], multiplicative preference relations [28, 36, 38], and linguistic preference relations [5, 26, 29, 47]. Sometimes, decision-makers have no self-confidence on the provided preference information because of time pressure and/or limited expertise regarding the problem domain. In these situations, decision-makers may provide their preference information in the form of incomplete preference relation, i.e., a preference relation with some of its elements missing [1, 22, 23, 32, 43, 46].

In order to deal with incomplete preference relations, Carmone [6] conducted an interesting Monte Carlo simulation to investigate the effect of reduced sets of pairwise comparisons in the Analytic Hierarchy Process (AHP). Herrera-Viedma et al. [24] proposed an additive-consistency based iterative procedure for estimating the missing values in a decision-maker's incomplete fuzzy preference relation using the provided preference values. Fedrizzi and Giove [19] proposed a completion method of incomplete preference relations by minimizing a measure of global inconsistency, thus obtaining an optimal preference relation from the point of view of consistency with respect to the available judgments. Jandová et al. [25] presented an interactive algorithm to compute interval weights for incomplete pairwise comparison matrices in large-dimensional problems, which was based on the sequential optimal choice of the pairwise comparisons to be performed and the concept of weak consistency. Ergu and Kou [18] proposed an induced bias matrix model to estimate the missing comparisons in a questionnaire survey while preserving the global consistency. Chen et al. [8] developed a procedure to solve the mixed problem of missing values and inconsistency with a single connecting path method. Triantaphyllou [42] formulated a linear programming formulation to estimate the missing values in incomplete preference relations. Büyüközkan and Çifçi [4] extended the quality function deployment methodology by introducing a new group decision-making approach that considers the incomplete information of decision-makers by means of the fuzzy set theory.

In an incomplete preference relation, two self-confidence levels can be stated: the decision-makers are self-confident when their preference values for the pairwise (x_i, x_j) is provided; otherwise, the decision-makers lack self-confidence. Recently, in an attempt to provide a more general theoretical context, Liu et al. [29] proposed the self-confident preference relations to allow decision-makers express multiple self-confidence levels when providing their preferences. Meanwhile, using the hesitancy degree of the reciprocal intuitionistic fuzzy preference relation, Ureña et al. [44] defined a concept of decision-maker's confidence to propose a new consistency and confidence-induced ordered weighted averaging operator in a group/multicriteria decision-making problem.

Although incomplete preference relations are very useful in decision-making and have been investigated intensively, there are still questions that need answering:

- (1) What are the conditions under which incomplete preference relations are better than complete preference relations?
- (2) Can self-confident preference relations improve the quality of decision-making?

In order to answer these questions, this paper presents a simulation-based investigation with the following features:

- (i) According to the bounded rationality principle [31, 37, 39], widely used as a basis to describe choice behaviors of decision-makers [3, 12], the approach of Triantaphyllou and Mann [41] is generalized to develop three assumptions, Assumptions 1-3 (Section 3 and Section 5), to generate, in the proposed simulation-based investigation, complete preference relations, incomplete preference relations, and self-confident preference relations, respectively.
- (ii) Based on Assumptions 1-3, Simulation Method I (Section 4) and Simulation Method II (Section 5) report on the design and results of the simulation experiments. The results show that incomplete preference relations outperform complete preference relations when the incomplete degree of preference relations is between 20% and 40% and the decision-maker is of medium rational degree. On the other hand, in most cases using self-confident preference relations improve the decision quality when compared with using incomplete preference relations.
- (iii) Finally, a sensitivity analysis (Section 6) on the key parameters in the simulation experiments is included to contribute to the robustness of the experiments' conclusions.

The mathematical mapping among different types of preference relations has been studied in Chen et al. [9], which allows the focus of study in this paper to be placed on multiplicative preference relations, and the proposed methods and results to be similarly applied to additive preference relations and linguistic preference relations via the corresponding transformation functions.

The remainder of the paper is organized as follows: Section 2 introduces preliminary knowledge needed later for the simulation-based investigation, which is based on the assumptions presented in Section 3. Section 4 develops Simulation Method I to compare the performance between incomplete preference relations and complete preference relations. Section 5 develops Simulation Method II to compare the performance between incomplete preference relations and self-confident preference relations. Section 6 reports on the sensitivity analysis on the key parameters in the simulation experiments. Finally, concluding remarks are drawn in Section 7.

2. Preliminaries

This section contains preliminary knowledge regarding multiplicative preference

relations, logarithmic least squares method, and an iterative method to estimate missing values in incomplete multiplicative preference relations that will provide a basis for the proposed simulation-based investigation.

2.1 Multiplicative Preference Relations and Logarithmic Least Squares Method

A multiplicative preference relation is defined as below.

Definition 1 [36]: A matrix $A = (a_{ij})_{n \times n}$ is called a multiplicative preference relation when $a_{ij} \cdot a_{ji} = 1$ and $a_{ij} > 0 \quad \forall i, j$, where a_{ij} indicates a ratio of the preference intensity of alternative x_i to that of alternative x_j .

In multiplicative preference relations, Saaty's scale of 17 numerical values is used:

$$\left\{ \frac{1}{i}, 1, i \right\}, \quad i = 2, 3, \dots, 9. \quad (1)$$

The logarithmic least squares method (LLSM) is most commonly used to derive a priority vector from multiplicative preference relations [13]. Let $w = (w_1, w_2, \dots, w_n)^T$ be the priority vector of multiplicative preference relation $A = (a_{ij})_{n \times n}$, where $w_i > 0 \quad (i = 1, 2, \dots, n)$ and $\sum_{i=1}^n w_i = 1$. The priority vector characterizes a consistent multiplicative preference relation [36], i.e. matrix $A = (a_{ij})_{n \times n}$ verifies $a_{ij} = a_{ik} \cdot a_{kj}, \quad \forall i, j, k$, when

$$a_{ij} = w_i / w_j, \quad i, j = 1, 2, \dots, n. \quad (2)$$

When a multiplicative preference relation is not consistent, the expression

$$e_{ij} = a_{ij} \times w_j / w_i, \quad i, j = 1, 2, \dots, n, \quad (3)$$

measures the error between the preference value a_{ij} and the corresponding consistent preference value built with priority vector w as per Eq. (2) [13, 36], which can be equivalently written as

$$\log e_{ij} = \log a_{ij} - \log w_i + \log w_j, \quad i, j = 1, 2, \dots, n. \quad (4)$$

Thus, the priority vector $w = (w_1, w_2, \dots, w_n)^T$ can be obtained by solving the following logarithmic least squares model [13]:

$$\begin{aligned} & \min \sum_{i=1}^n \sum_{j=1}^n (\log e_{ij})^2 \\ & s.t. \begin{cases} \sum_{i=1}^n w_i = 1, \\ w_i > 0, \quad i = 1, \dots, n. \end{cases} \end{aligned} \quad (5)$$

Crawford and Williams [13] showed that the optimal solution to the above model can be expressed as the row geometric mean of $A = (a_{ij})_{n \times n}$:

$$w_i = \frac{\left(\prod_{j=1}^n a_{ij}\right)^{1/n}}{\sum_{k=1}^n \left(\prod_{j=1}^n a_{kj}\right)^{1/n}}, \quad i=1, 2, \dots, n. \quad (6)$$

2.2 Incomplete Multiplicative Preference Relations and an Iterative Method to Estimate Missing Values

A complete preference relation requires a decision-maker to provide $n(n-1)/2$ preference values. In real decision-making problems, decision-makers may provide their preference information in the form of incomplete preference relation, i.e., with some of its elements missing. In the following, we give the definition of incomplete multiplicative preference relations.

Definition 2: A matrix $A = (a_{ij})_{n \times n}$ is called an incomplete multiplicative preference relation, if some of its elements are missing and the provided elements satisfy $a_{ij} \cdot a_{ji} = 1$ and $a_{ij} > 0 \quad \forall i, j$.

Similarly to the method developed by Herrera-Viedma et al. [24] for estimating missing values in an incomplete additive preference relation, the missing values of an incomplete multiplicative preference relation can be estimated as follows.

I. The following sets are defined:

$$\begin{aligned} V &= \{(i, j) \mid i, j \in \{1, \dots, n\} \wedge i \neq j\} \\ MV &= \{(i, j) \in V \mid a_{ij} \text{ is unknown}\} \\ EV &= V \setminus MV \\ H_{ik}^1 &= \{j \neq i, k \mid (i, j), (j, k) \in EV\} \\ H_{ik}^2 &= \{j \neq i, k \mid (j, i), (j, k) \in EV\} \\ H_{ik}^3 &= \{j \neq i, k \mid (i, j), (k, j) \in EV\}, \end{aligned} \quad (7)$$

where MV is the set of incomparable pairs of alternatives; EV is the set of pairs of alternatives for which the decision-maker provides preference values; H_{ik}^1 , H_{ik}^2 , and H_{ik}^3 are the sets of intermediate alternatives x_j ($j \neq i, k$) that can be used to estimate the preference value a_{ik} ($i \neq k$) using $ca_{ik}^{j1} = a_{ij} \cdot a_{jk}$, $ca_{ik}^{j2} = a_{jk} / a_{ji}$, and $ca_{ik}^{j3} = a_{ij} / a_{kj}$, respectively.

II. The subset of missing values MV that can be estimated in step h is denoted by EMV_h and defined as follows,

$$EMV_h = \left\{ (i, k) \in MV \setminus \bigcup_{l=0}^{h-1} EMV_l \mid i \neq k \wedge \exists j \in \{H_{ik}^{h1} \cup H_{ik}^{h2} \cup H_{ik}^{h3}\} \right\}, \quad (8)$$

with

$$\begin{aligned}
H_{ik}^{h1} &= \left\{ j \mid (i, j), (j, k) \in \left\{ EV \bigcup_{l=0}^{h-1} EMV_l \right\} \right\} \\
H_{ik}^{h2} &= \left\{ j \mid (j, i), (j, k) \in \left\{ EV \bigcup_{l=0}^{h-1} EMV_l \right\} \right\} \\
H_{ik}^{h3} &= \left\{ j \mid (i, j), (k, j) \in \left\{ EV \bigcup_{l=0}^{h-1} EMV_l \right\} \right\}
\end{aligned} \tag{9}$$

with $EMV_0 = \phi$ (by definition). The iterative procedure will stop when $EMV_{\maxIter} = \phi$ with $\maxIter > 0$. Moreover, if $\bigcup_{l=0}^{\maxIter} EMV_l = MV$, then all missing values are estimated, and consequently, the procedure is said to be successful in completing the incomplete multiplicative preference relation. The complete iterative estimation procedure is described in Algorithm 1.

Algorithm 1. Iterative estimation procedure.

Input: Incomplete multiplicative preference relation A .

Output: Complete preference relation A^C .

Step 1: The set $EMV_0 = \phi$ and $h = 1$

Step 2: While $EMV_h \neq \phi$

$h = h + 1$;

For every $(i, k) \in EMV_h$, calculate $ca_{ik} = (ca_{ik}^1 \times ca_{ik}^2 \times ca_{ik}^3)^{(1/\kappa)}$, where

(a) If $\#H_{ik}^{h1} = 0$, let $ca_{ik}^1 = 0$; else if $\#H_{ik}^{h1} \neq 0$, calculate $ca_{ik}^1 = (\prod_{j \in H_{ik}^{h1}} ca_{ik}^{j1})^{(1/\#H_{ik}^{h1})}$;

(b) If $\#H_{ik}^{h2} = 0$, let $ca_{ik}^2 = 0$; else if $\#H_{ik}^{h2} \neq 0$, calculate $ca_{ik}^2 = (\prod_{j \in H_{ik}^{h2}} ca_{ik}^{j2})^{(1/\#H_{ik}^{h2})}$;

(c) If $\#H_{ik}^{h3} = 0$, let $ca_{ik}^3 = 0$; else if $\#H_{ik}^{h3} \neq 0$, calculate $ca_{ik}^3 = (\prod_{j \in H_{ik}^{h3}} ca_{ik}^{j3})^{(1/\#H_{ik}^{h3})}$.

Let $a_{ik} = ca_{ik}$.

End while

Step 3: If $a_{ik} < 1/9$ or $a_{ik} > 9$, the median function is applied to normalize the expression domains

$$f(a_{ik}) = \text{med} \left(\frac{1}{9}, a_{ik}, 9 \right).$$

Step 4: Let $A^C = A$. Output A^C .

A sufficient condition for Algorithm 1 to estimate all missing values is given in the following result:

Proposition 1 [49]: The incomplete multiplicative preference relation can be completed by the Algorithm 1 iterative method if a set of $n-1$ of its non-leading diagonal preference values, where each alternative is compared at least once, is known.

The derivation of the priority vector of an incomplete multiplicative preference relation requires first the application of Algorithm 1 for its completion followed by the application of

the LLSM method.

Note. There are two different methods to estimate the missing values of an incomplete preference relation: iterative methods [23], and optimization methods [19, 42]. Because both the iterative and optimization methods are based on the use of consistency criteria, in many cases the corresponding outputs of the two methods are similar [10, 43]. Without loss of generality, in this paper the iterative method is used to estimate the missing values of incomplete multiplicative preference relations.

3. Assumptions of the Simulation Experiments

This section sets out the assumptions of the proposed simulation-based investigation.

3.1 Approximated Preference Relation of a Priority Vector

Let $w = (w_1, w_2, \dots, w_n)^T$ be a true priority vector of the set of alternatives $X = \{x_1, x_2, \dots, x_n\}$ and $w = (w_{ij})_{n \times n} = \left(\frac{w_i}{w_j} \right)_{n \times n}$ be the characteristic matrix associated with it. Although the w_{ij} values in $w = (w_{ij})_{n \times n} = \left(\frac{w_i}{w_j} \right)_{n \times n}$ belong to a continuous domain of real numbers, it is noted above that decision-makers can only provide a preference relation based on a discrete scale of values (Saaty's scale). Thus, before presenting the proposed simulation-based investigation, Question 1 below is discussed.

Question 1: For a true priority vector $w = (w_1, w_2, \dots, w_n)^T$ of the set of alternatives $X = \{x_1, x_2, \dots, x_n\}$, how will rational decision-makers express their preference relation to approximate the true priority vector w ?

Triantaphyllou and Mann [41] proposed an elegant, concise and interesting method by assuming that the decision-maker is always able to choose the closest value from Saaty's scale to $w = (w_{ij})_{n \times n} = \left(\frac{w_i}{w_j} \right)_{n \times n}$. Specifically, for $i \neq j$, when $w_{ij} \geq 1$, the decision-maker chooses the value from $\{1, 2, \dots, 9\}$ closest to w_{ij} ; while for $w_{ij} < 1$, its closest value from the reciprocal set of values to $\{1, 2, \dots, 9\}$ is chosen. However, in real life it is difficult for decision-makers to choose the closest value to approximate w_{ij} in all cases. Decision-makers are prone to errors and biases according to the bounded rationality principle [12, 31, 37, 39], thus the method of Triantaphyllou and Mann [41] needs to be generalized or extended to cover these situations. Let $\{w_{\sigma(1)}, w_{\sigma(2)}, \dots, w_{\sigma(9)}\}$ be a permutation of $\{1, 2, \dots, 9\}$, where $w_{\sigma(g)}$ is the g -th closest value of w_{ij} . It is assumed that p_g is the probability of choosing $w_{\sigma(g)}$ to approximate w_{ij} , and then we can get a probability vector $P = (p_1, p_2, \dots, p_9)$.

According to the bounded rationality principle, the probability vector should satisfy $p_i \geq 0$, $p_i \geq p_{i+1}$ and $\sum_{i=1}^9 p_i = 1$. Here, we formally propose the following Assumption 1 that will be used in the proposed simulation-based investigation.

Assumption 1: Let $w = (w_1, w_2, \dots, w_n)^T$ be a true priority vector. For $i \neq j$, when $w_{ij} \geq 1$, a rational decision-maker will express the preference value $w_{\sigma(g)}$ with the probability p_g ($g = 1, 2, \dots, 9$) to approximate w_{ij} . This is symbolized by $\{w_{\sigma(1)}/p_1, w_{\sigma(2)}/p_2, \dots, w_{\sigma(9)}/p_9\}$, where $p_i \geq 0$, $p_i \geq p_{i+1}$ and $\sum_{i=1}^9 p_i = 1$; when $w_{ij} < 1$, the reciprocal property is used.

Clearly, the method of Triantaphyllou and Mann is obtained by setting $P = (p_1, p_2, \dots, p_9) = (1, 0, 0, \dots, 0)$.

Example 1: Let $w = (0.23, 0.50, 0.10, 0.17)^T$ be a true priority vector with characteristic matrix

$$W = (w_{ij})_{4 \times 4} = \begin{pmatrix} \frac{w_i}{w_j} \end{pmatrix}_{4 \times 4} = \begin{pmatrix} 1.000 & 0.460 & 2.300 & 1.353 \\ 2.174 & 1.000 & 5.000 & 2.941 \\ 0.435 & 0.200 & 1.000 & 0.588 \\ 0.739 & 0.340 & 1.700 & 1.000 \end{pmatrix}.$$

If $P = (1, 0, 0, 0, 0, 0, 0, 0, 0)$, Assumption 1 reduces to the method of Triantaphyllou and Mann [41], and produces the following approximated multiplicative preference relation

$$\overline{W} = \begin{pmatrix} 1 & 0.5 & 2 & 1 \\ 2 & 1 & 5 & 3 \\ 0.5 & 0.2 & 1 & 0.5 \\ 1 & 0.33 & 2 & 1 \end{pmatrix}.$$

If $P = (0.9, 0.1, 0, 0, 0, 0, 0, 0, 0)$, then the decision-maker will express the preference value $w_{\sigma(1)}$ with probability 0.9 and the preference value $w_{\sigma(2)}$ with probability 0.1 to approximate w_{ij} . In this case, the approximated preference value \overline{w}_{ij} will be $\{w_{\sigma(1)}/0.9, w_{\sigma(2)}/0.1\}$ for the values $w_{ij} \geq 1$,

$$\overline{W} = \begin{pmatrix} 1 & - & \{2/0.9, 3/0.1\} & \{1/0.9, 2/0.1\} \\ \{2/0.9, 3/0.1\} & 1 & \{5/0.9, 4/0.1\} & \{3/0.9, 2/0.1\} \\ - & - & 1 & - \\ - & - & \{2/0.9, 1/0.1\} & 1 \end{pmatrix}.$$

Obviously, different probability vectors can be used to reflect different rational degrees

of decision-makers. In this paper we define the rational degree index as follows:

Definition 3: Let $P = (p_1, p_2, \dots, p_9)$ be a probability vector used in Assumption 1. The rational degree index (RDI) of P is

$$\text{RDI}(P) = \sum_{i=1}^9 i \times p_i. \quad (10)$$

The smaller the value of $\text{RDI}(P)$, the better the rational degree of P . For example, let $P_1 = (1, 0, 0, 0, 0, 0, 0, 0, 0)$, $P_2 = (0.9, 0.1, 0, 0, 0, 0, 0, 0, 0)$, and $P_3 = (0.8, 0.1, 0.1, 0, 0, 0, 0, 0, 0)$ be three probability vectors. Based on Eq. (10), we have $\text{RDI}(P_1)=1$, $\text{RDI}(P_2)=1.1$, and $\text{RDI}(P_3)=1.3$. Thus, the rational degree index of P_1 is better than that of P_2 , which is better than that of P_3 .

3.2 Formation of an Incomplete Preference Relation

Let $A^C = (a_{ij}^C)_{n \times n}$ be a complete preference relation randomly generated over the set of alternatives $X = \{x_1, x_2, \dots, x_n\}$. Before constructing incomplete preference relations in the proposed simulation-based investigation, Question 2 below is discussed.

Question 2: Let $A^C = (a_{ij}^C)_{n \times n}$ be a complete preference relation. How will a rational decision-makers express their incomplete preference relation with k missing pairwise comparison values to approximate $A^C = (a_{ij}^C)_{n \times n}$?

Let $w^C = (w_1^C, w_2^C, \dots, w_n^C)^T$ be the priority vector derived from $A^C = (a_{ij}^C)_{n \times n}$ by LLSM [13], and let $\varepsilon^C = (\varepsilon_{ij}^C)_{n \times n}$ be the error matrix associated with $A^C = (a_{ij}^C)_{n \times n}$, where $\varepsilon_{ij}^C = (\log a_{ij}^C - \log w_i^C + \log w_j^C)^2$ measures the error degree of preference value a_{ij}^C . The larger the value ε_{ij}^C , the larger the error degree of preference value a_{ij}^C . It is natural that a rational decision-maker will delete the k preference values with largest error degrees to approximate $A^C = (a_{ij}^C)_{n \times n}$. Thus, we formally propose the following Assumption 2 that will be used in the proposed simulation-based investigation.

Assumption 2: Let $A^C = (a_{ij}^C)_{n \times n}$ be a complete preference relation. When a rational decision-maker expresses the incomplete preference relation $A^{I,k}$ with k pairwise comparison missing values, the k preference values with largest error degrees will be deleted in $A^C = (a_{ij}^C)_{n \times n}$.

Example 2: Let

$$A^C = \begin{pmatrix} 1 & 0.5 & 2 & 1 \\ 2 & 1 & 5 & 3 \\ 0.5 & 0.2 & 1 & 0.5 \\ 1 & 0.33 & 2 & 1 \end{pmatrix}$$

be a complete preference relation and $w_c = (0.212, 0.496, 0.100, 0.192)^T$ its priority vector obtained by LLSM. Then its associated error matrix is

$$\varepsilon^C = \begin{pmatrix} 0 & 0.0046 & 0.0006 & 0.0019 \\ 0.0046 & 0 & 0 & 0.0042 \\ 0.0006 & 0 & 0 & 0.0003 \\ 0.0019 & 0.0042 & 0.0003 & 0 \end{pmatrix}.$$

Applying [Assumption 2](#) with $k=1, 2$ and 3 results in the following incomplete preference relations, respectively.

$$A^{I,1} = \begin{pmatrix} 1 & - & 2 & 1 \\ - & 1 & 5 & 3 \\ 0.5 & 0.2 & 1 & 0.5 \\ 1 & 0.33 & 2 & 1 \end{pmatrix}, \quad A^{I,2} = \begin{pmatrix} 1 & - & 2 & 1 \\ - & 1 & 5 & - \\ 0.5 & 0.2 & 1 & 0.5 \\ 1 & - & 2 & 1 \end{pmatrix}, \quad \text{and} \quad A^{I,3} = \begin{pmatrix} 1 & - & 2 & - \\ - & 1 & 5 & - \\ 0.5 & 0.2 & 1 & 0.5 \\ - & - & 2 & 1 \end{pmatrix}.$$

4. Simulation Experiments: Incomplete vs. Complete

In this section, Simulation Method I is proposed to compare the use of incomplete preference relations and complete preference relations in decision making in terms of accuracy performance measure based on the Manhattan distance function. Afterwards, the experimental results obtained with Simulation Method I are presented and analyzed.

4.1 Simulation Method I

Let $w = (w_1, w_2, \dots, w_n)^T$ be the true priority vector of the set of alternatives $X = \{x_1, x_2, \dots, x_n\}$; $W = (w_{ij})_{n \times n}$ its the characteristic matrix; $\{w_{\sigma(1)}, w_{\sigma(2)}, \dots, w_{\sigma(9)}\}$ a permutation of $\{1, 2, \dots, 9\}$, where $w_{\sigma(g)}$ is the g -th closest value of w_{ij} ($i \neq j$); and $P = (p_1, p_2, \dots, p_9)$ a probability vector, such that $p_i \geq 0$, $p_i \geq p_{i+1}$ and $\sum_{i=1}^9 p_i = 1$.

Using [Assumption 1](#), a decision-maker will approximate w_{ij} with the preference value $w_{\sigma(g)}$ with probability p_g ($g=1, 2, \dots, 9$). In this way, a complete preference relation $A^C = (a_{ij}^C)_{n \times n}$ can be randomly generated with the preference value a_{ij}^C being $w_{\sigma(g)}$ with probability p_g ($g=1, 2, \dots, 9$). Let w^C be the priority vector of $A^C = (a_{ij}^C)_{n \times n}$ obtained by LLSM.

Saaty [\[35\]](#) presented an approach to calculate the deviation between the true priority vector w and the derived priority vector w^C . Inspired by Saaty's work, herein the following

Manhattan distance function is considered to measure how well the priority vector derived from the complete preference relation is able to approximate the true priority vector

$$\text{DCT} = \sum_{i=1}^n |w_i^C - w_i|. \quad (11)$$

The smaller the DCT value is, the better the performance of the complete preference relation A^C .

Based on [Assumption 2](#), when a rational decision-maker expresses the incomplete preference relation $A^{I,k}$ with k pairwise comparison missing values, the largest k preference values with largest associated error degrees will be deleted from $A^C = (a_{ij}^C)_{n \times n}$. Using this method we can generate the incomplete preference relation $A^{I,k}$. If $A^{I,k}$ satisfies [Proposition 1](#), then we can estimate all missing values by the provided Algorithm 1. Then, we can derive the priority vector $w^{I,k}$ from $A^{I,k}$ by LLSM after completion. The following Manhattan distance function is considered to measure how well the priority vector generated by the incomplete preference relation approximates the true priority vector :

$$\text{DIT} = \sum_{i=1}^n |w_i^{I,k} - w_i|. \quad (12)$$

The smaller the DIT value is, the better the performance of incomplete preference relation $A^{I,k}$.

Simulation Method I is proposed to compare the respective performance value of complete preference relations and incomplete preference relations as measured by Based on and , respectively. The Simulation I algorithm is based on the following: first a true priority vector w is randomly generated, followed by the derivation of its characteristic matrix W . Then, based on [Assumption 1](#), a complete preference relation A^C is randomly generated and its priority vector w^C computed. Based on [Assumption 2](#), the incomplete preference relation $A^{I,k}$ is generated and its priority vector $w^{I,k}$ computed. Finally, DCT and DIT are derived and used to find out when incomplete preference relations outperform complete preference relations. Next, we describe Simulation Method I in its algorithmic form.

Simulation I Algorithm. Simulation Method I to compare performance values of incomplete preference relations and complete preference relations.

Input: The dimension of the preference relation n , the probability vector P , the number of pairwise comparison missing values k .

Output: R and S .

Step 1: Generate true priority vector $w = (w_1, w_2, \dots, w_n)^T$, where the values w_i ($i = 1, 2, \dots, n$) are uniformly randomly selected from $[0, 1]$. Then, normalize w so that $\sum_{i=1}^n w_i = 1$. Let

$W = (w_{ij})_{n \times n} = \left(\frac{w_i}{w_j} \right)_{n \times n}$ be the characteristic matrix associated with the true priority vector w .

Step 2: Let $\{w_{ij, \sigma(1)}, w_{ij, \sigma(2)}, \dots, w_{ij, \sigma(9)}\}$ be a permutation of $\{1, 2, \dots, 9\}$, where $w_{ij, \sigma(g)}$ is the g -th closest value of w_{ij} . Based on [Assumption 1](#), randomly generate a complete preference relation $A^C = (a_{ij}^C)_{n \times n}$, where the preference value a_{ij}^C is $w_{ij, \sigma(g)}$ with probability p_g ($g = 1, 2, \dots, 9$). Let w^C be the priority vector of A^C obtained by LLSM.

Step 3: Based on [Assumption 2](#), delete the k preference values with largest associated error degrees in A^C to generate the incomplete preference relations $A^{I,k}$. If $A^{I,k}$ satisfies [Proposition 1](#), then estimate all missing values using Algorithm 1. Let $w^{I,k}$ be the priority vector of $A^{I,k}$ obtained by LLSM after completion. If $A^{I,k}$ does not satisfy [Proposition 1](#), go to Step 1.

Step 4: Calculate $DCT = \sum_{i=1}^n |w_i^C - w_i|$ and $DIT = \sum_{i=1}^n |w_i^{I,k} - w_i|$. Let $R = \frac{DCT}{DIT}$. If $DIT < DCT$, let $S = 1$. Otherwise, let $S = 0$. Output R and S .

In Simulation Method I, the difference between $w^{I,k}$ and w is smaller than the difference between w^C and w when $R > 1$ or $S = 1$, which means that the priority vector $w^{I,k}$ approximates the true priority vector w better than the priority vector w^C . Meanwhile, the priority vector w^C approximates the true priority vector w better than the priority vector $w^{I,k}$ when $R < 1$ or $S = 0$.

Remark 1. In Step 3 of Simulation Method I, if the incomplete preference relation $A^{I,k}$ satisfies [Proposition 1](#), the $A^{I,k}$ can be completed by the given iterative estimation procedure (Algorithm 1) and the Simulation algorithm I will continue. Otherwise, it is impossible to derive a priority vector from $A^{I,k}$, and thus a new incomplete preference relation is to be created in a new round. Based on [Proposition 1](#), it is required to set $k \leq \left(\frac{n(n-1)}{2} - (n-1) \right)$ when using Simulation Method I.

4.2 Experimental Results: Incomplete vs. Complete

The experiment settings based on Simulation Method I are as follows.

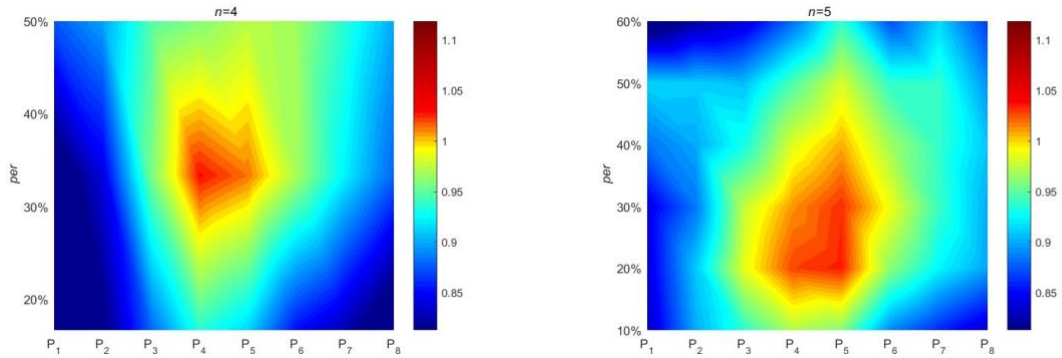
(1) Eight (8) probability vectors are used to reflect different rational degree of decision-makers. Based on [Definition 3](#), the RDI from P_1 to P_8 is gradually decreased.

$$\begin{aligned}
P_1 &= (1, 0, 0, 0, 0, 0, 0, 0) \\
P_2 &= (0.9, 0.1, 0, 0, 0, 0, 0, 0) \\
P_3 &= (0.8, 0.1, 0.1, 0, 0, 0, 0, 0) \\
P_4 &= (0.7, 0.1, 0.1, 0.1, 0, 0, 0, 0) \\
P_5 &= (0.6, 0.1, 0.1, 0.1, 0.1, 0, 0, 0) \\
P_6 &= (0.5, 0.1, 0.1, 0.1, 0.1, 0.1, 0, 0) \\
P_7 &= (0.4, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0) \\
P_8 &= (0.3, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1)
\end{aligned} \tag{13}$$

(2) In the simulation, the initial priority values w_i ($i=1, 2, \dots, n$) are the exact numbers that are randomly selected from $[0, 1]$. In addition, let $per = \frac{2 \times k}{n(n-1)} \times 100\%$ measure the incompleteness degree in a preference relation with k pairwise comparison missing values. For example, when setting the size of preference relations at $n=5$ and the number of pairwise comparison missing values at $k=2$, the incompleteness degree will be $per = 20\%$. Since $k \leq \left(\frac{n(n-1)}{2} - (n-1) \right)$, we have that $per \leq \frac{n-2}{n} \times 100\%$.

Different input parameters n , P and k are set, and Simulation Method I is run 1000 times to obtain the average values of R and S under different parameters. The larger the average values of R and S , the better the priority vector $w^{I,k}$ approximates the true priority vector w . Meanwhile, the smaller the average values of R and S , the better the priority vector w^C approximates the true priority vector w . When the average $R > 1$ and $S > 50\%$, the experimental results indicate that incomplete preference relations outperform the complete preference relations. On the other hand, when the average $R < 1$ and $S < 50\%$ the complete preference relations outperform the incomplete preference relations.

The experimental results of average values of R and S under different parameters are pictured in Figs. 1 and 2, respectively. The following observations can be drawn.



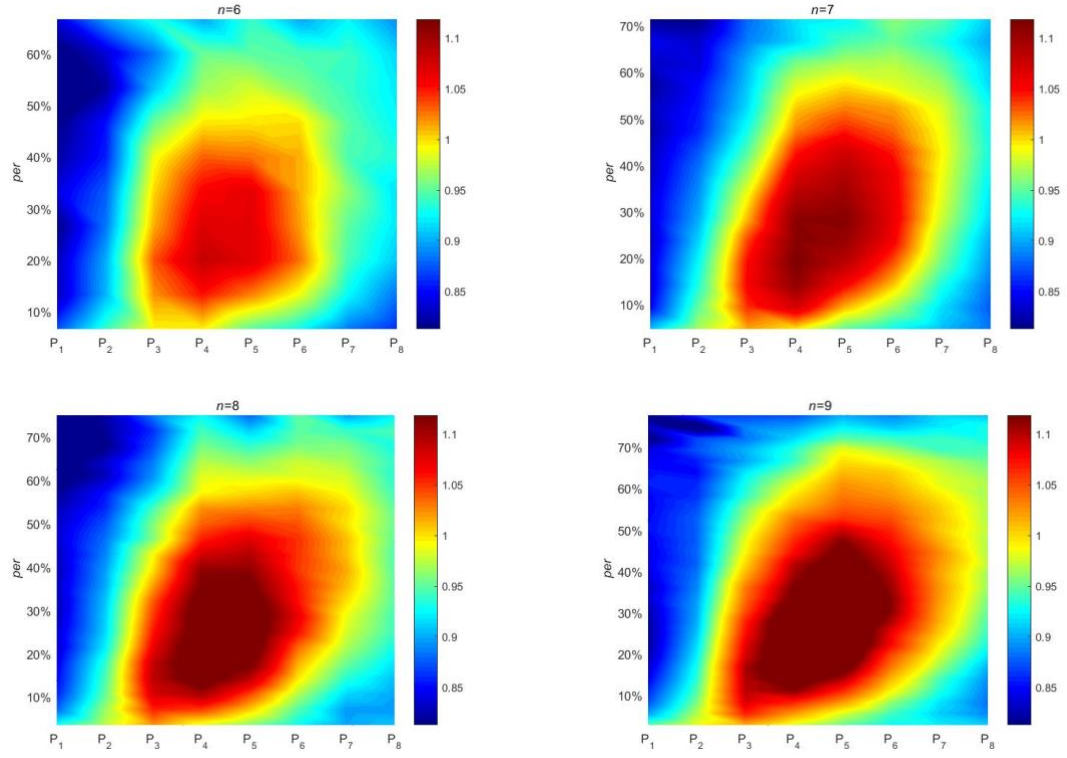
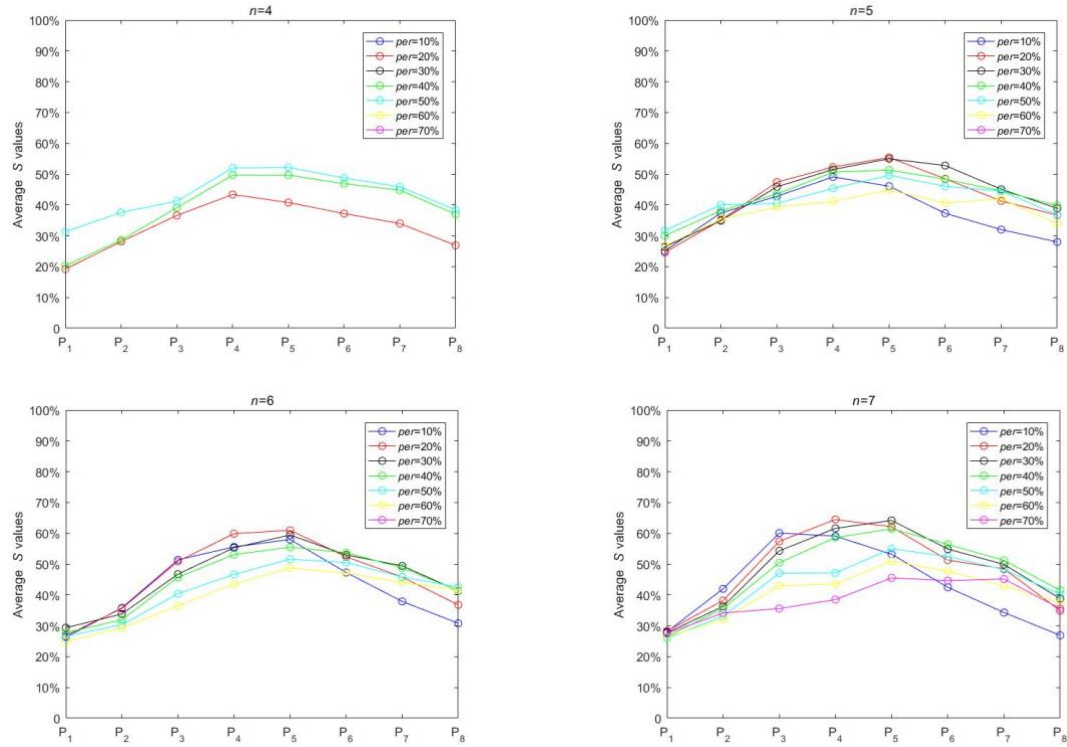


Fig.1. Average values of R under different n , P , and per values.



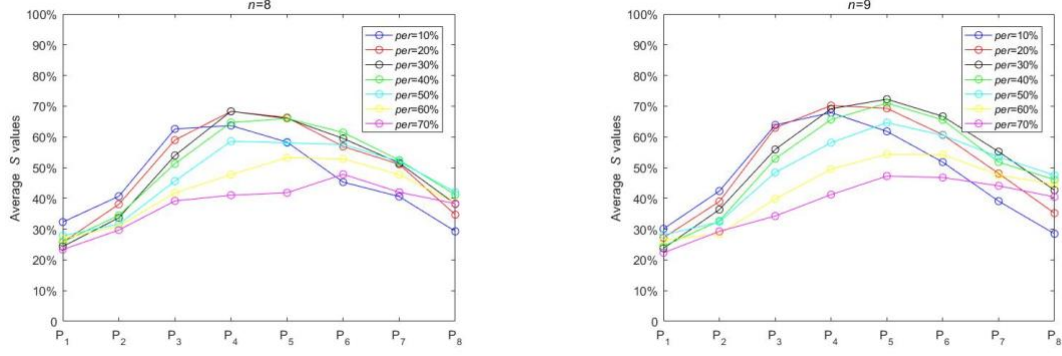


Fig.2. Average values of S under different n , P , and per values.

(1) Fig. 1 illustrates how the average values of R change with respect to the size of the preference relation, the rational degree, and the incompleteness degree. From Fig. 1 the following two observations are drawn:

- (i) When the rational degree of decision-makers ranges from P_3 to P_6 and the incompleteness degree of the preference relation is between 20% and 40%, the average values of R are greater than 1 (the red part in Fig. 1) and therefore the incomplete preference relation outperforms the complete preference relation.
- (ii) When the rational degree is high (e.g., P_1 and P_2) or low (e.g., P_7 and P_8), and the incompleteness degree is greater than 60% or lower than 10%, the average values of R are less than 1 (the non-red part in Fig. 1) and therefore the complete preference relations outperforms the incomplete preference relations.

(2) Fig. 2 further helps us understand the impact of the rational degree and the incompleteness degree of a preference relation on the quality of the priority vectors. When the rational degree is ranges from P_3 to P_6 and the incompleteness degree is between 20% to 40%, the average values of S are higher than 50%, i.e. the number of times that the incomplete preference relation outperforms the complete preference relation is higher than the number of times the opposite. In summary, incomplete preference relations have better performance when the incompleteness degree of preference relations is between 20% and 40%, and the decision-maker is of medium rational degree.

(3) The above conclusions are more evident for high values of the dimension of preference relations. As the dimension of preference relations increases, the area of red in Fig. 1 increases while the average value of S increases as per Fig. 2. As a result, when the incompleteness degree and the rational degree are both medium, the use of incomplete preference relations will further improve the quality of the priority vectors with the increase of the dimension of preference relations.

Although incomplete preference relations have been widely used, Simulation Method I provides evidence that indicates their use in all cases not to be the best course of action to

achieve the best quality decision. The use of incomplete preference relations in decision-making can be better supported by using the observations drawn above from the average values of both R and S .

Remark 2. We find that the rational degree and incompleteness degree have an important impact on the performance of incomplete preference relations, which means that the efficient use of incomplete preference relations depends on the estimation of the rational degree of a decision-maker. However, we argue that it is a challenging task to estimate the rational degree of a decision-maker in real life. Instead, in the next section we introduce the concept of self-confident preference relations, in which the decision-maker is required to express multiple self-confidence levels over their pairwise comparison preference values.

5. Simulation Experiments: Incomplete vs. Self-confident

In this section, self-confident preference relations are defined. Simulation Method II is proposed to compare the performances of incomplete preference relations and self-confident preference relations. It will be shown that self-confident preference relations generally outperform incomplete preference relations.

5.1 Self-confident Preference Relations and Extended Logarithmic Least Squares Method

Rating scales are widely used in decision-making processes. A rating scale $\{1, 2, 3, \dots, N\}$ refers to an N -point scale in which people have N rating options: “1= extremely poor ... N = extremely good” [2, 11, 17, 21]. Let $S^{SL} = \{1, 2, \dots, N\}$ be an N -point rating scale used by decision-makers to characterize their self-confidence levels over the preference relation values they provide. The higher the value in S^{SL} , the higher the self-confidence level. Setting different N values results in different rating scales. Inspired by Liu et al. [29], we define self-confident preference relations as follows.

Definition 3: A matrix $A^* = ((a_{ij}^*, s_{ij}^*))_{n \times n}$ is called a self-confident preference relation if its elements have two components: the first component, a_{ij}^* , represents the preference degree or intensity of the alternative x_i over the alternative x_j , and the second component, $s_{ij}^* \in S^{SL}$, represents the self-confidence level associated with the preference value a_{ij}^* . The following conditions are assumed: $a_{ij}^* \cdot a_{ji}^* = 1$, $s_{ij}^* = s_{ji}^*$, and $s_{ii}^* = N$ for $i, j = 1, 2, \dots, n$.

In order to derive the priority vector of a self-confident preference relation, the LLSM is extended by adding the self-confidence values s_{ij}^* as coefficients in the model as follows:

$$\min \sum_{i=1}^n \sum_{j=1}^n s_{ij}^* \times (\log a_{ij}^* - \log w_i^* + \log w_j^*)^2$$

$$s.t. \begin{cases} \sum_{i=1}^n w_i^* = 1, \\ w_i^* > 0, \quad i = 1, \dots, n. \end{cases} \quad (14)$$

The self-confidence value s_{ij}^* in model (14) determines the magnification of error degree, with higher values of s_{ij}^* indicating a higher magnification of error degree. In this paper, model (14) is successfully solved using the MATLAB nonlinear optimization function “fmincon”.

As mentioned before, there is always at least two self-confidence levels in an incomplete preference relation in a broad sense: the decision-makers have self-confidence when making a pairwise comparison between alternatives x_i and x_j when they provides their preference values for such pair of alternatives (x_i, x_j) ; otherwise, preference values are missing because the decision-makers completely lack self-confidence. In a self-confident preference relation, multiple self-confidence levels may be used according to the rating scale $S^{SL} = \{1, 2, \dots, N\}$. We used Simulation Method II to compare the performances of incomplete preference relations and self-confident preference relations.

5.2 Simulation Method

Let $A^C = (a_{ij}^C)_{n \times n}$ be a complete preference relation over the set of alternatives $X = \{x_1, x_2, \dots, x_n\}$. Before presenting the simulation method for self-confident preference relations, the following Question 3 below is discussed.

Question 3: Given a rating scale $S^{SL} = \{1, 2, \dots, N\}$, how will rational decision-makers express self-confidence levels associated with their preference values a_{ij}^C in a complete preference relation $A^C = (a_{ij}^C)_{n \times n}$?

Let $\mathcal{E}^C = (\mathcal{E}_{ij}^C)_{n \times n}$ be the error matrix associated with $A^C = (a_{ij}^C)_{n \times n}$, as previously defined: $\mathcal{E}_{ij}^C = (\log a_{ij}^C - \log w_i^C + \log w_j^C)^2$. Let $\left\{ \mathcal{E}_{\sigma(1)}^C, \mathcal{E}_{\sigma(2)}^C, \dots, \mathcal{E}_{\sigma(\frac{n(n-1)}{2})}^C \right\}$ be a permutation of the upper triangular error values $\{\mathcal{E}_{ij}^C \mid i=1, 2, \dots, n; j=i+1, \dots, n\}$, where $\mathcal{E}_{\sigma(k)}^C$ is the k -th largest value in $\{\mathcal{E}_{ij}^C \mid i=1, 2, \dots, n; j=i+1, \dots, n\}$. It is assumed that E_1, E_2, \dots, E_N be subsets of $\{\mathcal{E}_{ij}^C \mid i=1, 2, \dots, n; j=i+1, \dots, n\}$, where

$$E_k = \left\{ \mathcal{E}_{\sigma\left(\left\lceil \frac{n(n-1)(k-1)}{2N} + 1 \right\rceil\right)}^C, \mathcal{E}_{\sigma\left(\left\lceil \frac{n(n-1)(k-1)}{2N} + 2 \right\rceil\right)}^C, \dots, \mathcal{E}_{\sigma\left(\left\lceil \frac{n(n-1)k}{2N} \right\rceil\right)}^C \right\}. \quad (15)$$

and the function $[x]$ denotes the largest integer which is smaller than x , i.e., the largest integer verifying $x - 1 \leq [x] \leq x$.

It is natural that a rational decision-maker will express a lower self-confident degree associated with the preference value a_{ij}^C with a higher error value ε_{ij}^C . Thus, the following [Assumption 3](#) is formally proposed for its use in Simulation Method II.

Assumption 3: Let $A^C = (a_{ij}^C)_{n \times n}$ be a complete preference relation, and let $S^{SL} = \{1, 2, \dots, N\}$ be a rating scale. If $\varepsilon_{ij}^C \in E_k$ ($i=1, 2, \dots, n$; $j=i+1, \dots, n$; $k=1, 2, \dots, N$), then a rational decision-maker expresses the self-confidence value $s_{ij} = k$ associated with the preference value a_{ij}^C , and $s_{ji} = s_{ij}$.

Example 3: Let

$$A^C = \begin{pmatrix} 1 & 0.5 & 2 & 1 \\ 2 & 1 & 5 & 3 \\ 0.5 & 0.2 & 1 & 0.5 \\ 1 & 0.33 & 2 & 1 \end{pmatrix}$$

be a complete preference relation and let

$$\varepsilon^C = \begin{pmatrix} 0 & 0.0046 & 0.0006 & 0.0019 \\ 0.0046 & 0 & 0 & 0.0042 \\ 0.0006 & 0 & 0 & 0.0003 \\ 0.0019 & 0.0042 & 0.0003 & 0 \end{pmatrix}$$

be its associated error matrix.

The permutation, from highest to smallest, of the upper triangular error values will be: $\{\varepsilon_{\sigma(1)}^C, \varepsilon_{\sigma(2)}^C, \varepsilon_{\sigma(3)}^C, \varepsilon_{\sigma(4)}^C, \varepsilon_{\sigma(5)}^C, \varepsilon_{\sigma(6)}^C\} = \{\varepsilon_{12}^C, \varepsilon_{24}^C, \varepsilon_{14}^C, \varepsilon_{13}^C, \varepsilon_{34}^C, \varepsilon_{23}^C\}$. Setting $S^{SL} = \{1, 2, 3\}$ results in $E_1 = \{\varepsilon_{12}^C, \varepsilon_{24}^C\}$, $E_2 = \{\varepsilon_{14}^C, \varepsilon_{13}^C\}$ and $E_3 = \{\varepsilon_{34}^C, \varepsilon_{23}^C\}$. Based on [Assumption 3](#), it is $s_{12}^* = s_{24}^* = 1$, $s_{14}^* = s_{13}^* = 2$, and $s_{34}^* = s_{23}^* = 3$. Thus, the following self-confident preference relation is obtained: $A^* = ((a_{ij}^*, s_{ij}^*))_{n \times n}$, where $a_{ij}^* = a_{ij}^C$, i.e.,

$$A^* = \begin{pmatrix} (1, 3) & (0.5, 1) & (2, 2) & (1, 2) \\ (2, 1) & (1, 3) & (5, 3) & (3, 1) \\ (0.5, 2) & (0.2, 3) & (1, 3) & (0.5, 3) \\ (1, 2) & (0.33, 1) & (2, 3) & (1, 3) \end{pmatrix}.$$

Let $w^* = (w_1^*, w_2^*, \dots, w_n^*)^T$ be the priority vector of $A^* = ((a_{ij}^*, s_{ij}^*))_{n \times n}$ obtained by the extended LLSM. The following Manhattan distance function is considered to evaluate how well the priority vector obtained from the self-confident preference relation is able to approximate the true priority vector :

$$\text{DST} = \sum_{i=1}^n |w_i^* - w_i|, \quad (16)$$

A small DST value indicates a high performance of the self-confident preference relation A^* .

Based on the use of DIT and DST (see Eqs. (12) and (16)), Simulation Method II is presented to compare incomplete preference relations against self-confident preference relations. Simulation II algorithm is based on the following: first a true priority vector w is randomly generated, followed by the derivation of its characteristic matrix W . Then, (1) based on [Assumption 1](#), a complete preference relation A^C is randomly generated and its error matrix ε^C is derived; (2) based on [Assumption 2](#), the incomplete preference relation $A^{I,k}$ is generated and its priority vector $w^{I,k}$ is derived; and (3) based on [Assumption 3](#), a self-confident preference relation A^* is generated and its priority vector w^* is derived. Finally, DIT and DST values are computed to compare the performances of incomplete preference relations and self-confident preference relations. Clearly, Simulation Method II is similar to Simulation Method I, and it can be formally described in algorithmic form below.

Simulation II Algorithm. Simulation Method II to compare the performances of incomplete preference relations and self-confident preference relations.

Input: The dimension of the preference relation n , the probability vector P , the number of pairwise comparison missing values k , and the rating scale $S^{SL} = \{1, 2, \dots, N\}$.

Output: R^* and S^* .

Step 1: Generate a true priority vector $w = (w_1, w_2, \dots, w_n)^T$, where the values w_i ($i = 1, 2, \dots, n$) are uniformly randomly generated from $[0, 1]$. Then, normalize w so that $\sum_{i=1}^n w_i = 1$. Let

$$W = (w_{ij})_{n \times n} = \left(\frac{w_i}{w_j} \right)_{n \times n} \text{ be the characteristic matrix associated with the true priority vector } w.$$

Step 2: Let $\{w_{ij, \sigma(1)}, w_{ij, \sigma(2)}, \dots, w_{ij, \sigma(9)}\}$ be a permutation of $\{1, 2, \dots, 9\}$, where $w_{ij, \sigma(g)}$ is the g -th closest value of w_{ij} . Based on [Assumption 1](#), randomly generate a complete preference relation $A^C = (a_{ij}^C)_{n \times n}$, where the preference value a_{ij}^C is $w_{ij, \sigma(g)}$ with probability p_g ($g = 1, 2, \dots, 9$). Let w^C be the priority vector of A^C obtained by LLSM.

Step 3: Based on [Assumption 2](#), delete the k preference values with largest associated error degrees in A^C to generate the incomplete preference relations $A^{I,k}$. If $A^{I,k}$ satisfies [Proposition 1](#), then estimate all missing values using Algorithm 1. Let $w^{I,k}$ be the priority vector of $A^{I,k}$ obtained by LLSM after completion. If $A^{I,k}$ does not satisfy [Proposition 1](#), go to Step 1.

Step 4: Based on [Assumption 3](#), generate the self-confidence value s_{ij} associated with the preference value a_{ij}^C and obtain the self-confident preference relation A^* . Let w^* be the priority vector of A^* obtained by extended LLSM.

Step 5: Compute $\text{DIT} = \sum_{i=1}^n |w_i^{I,k} - w_i|$ and $\text{DST} = \sum_{i=1}^n |w_i^* - w_i|$. Let $R^* = \frac{\text{DIT}}{\text{DST}}$. If $\text{DST} < \text{DIT}$, let $S^* = 1$. Otherwise, let $S^* = 0$. Output R^* and S^* .

In Simulation Method II, the difference between w^* and w is lower than the difference between $w^{I,k}$ and w when $R^* > 1$ or $S^* = 1$, which means that the priority vector w^* approximates the true priority vector w better than the priority vector $w^{I,k}$. Meanwhile, the priority vector $w^{I,k}$ approximates the true priority vector w better than the priority vector w^* when $R^* < 1$ or $S^* = 0$.

5.3 Experimental Results: Incomplete vs. Self-confident

The experiment settings based on Simulation Method II are as follows.

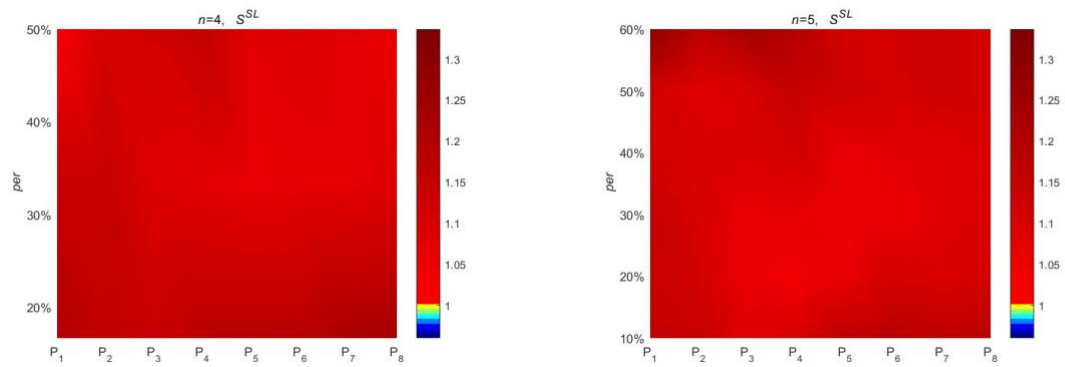
(1) The set $S^{SL} = \{1, 2, 3, 4, 5\}$ is used to characterize the self-confidence levels over the preference values.

(2) In the simulation, the initial priority values w_i ($i = 1, 2, \dots, n$) are the numbers randomly generated from $[0, 1]$. In addition, as in Simulation Method I, we let

$$per = \frac{2 \times k}{n(n-1)} \times 100\% \quad \text{because when } k \leq \left(\frac{n(n-1)}{2} - (n-1) \right), \quad per \leq \frac{n-2}{n} \times 100\% .$$

Different input parameters n , P and k are set, and Simulation Method II is run 1000 times to obtain the average values of R^* and S^* under different parameters. The higher the average values of R^* and S^* , the better the priority vector w^* approximates the true priority vector w . Meanwhile, the smaller the average values of R^* and S^* , the better the priority vector $w^{I,k}$ approximates the true priority vector w . When $R^* > 1$ and $S^* > 50\%$, the self-confident preference relations outperform incomplete preference relations. On the other hand, when $R^* < 1$ and $S^* < 50\%$, the incomplete preference relations outperform self-confident preference relations.

The experimental results obtained for the average values of R^* and S^* under different parameters are pictured in Figs. 3 and 4, respectively. The following observations can be drawn:



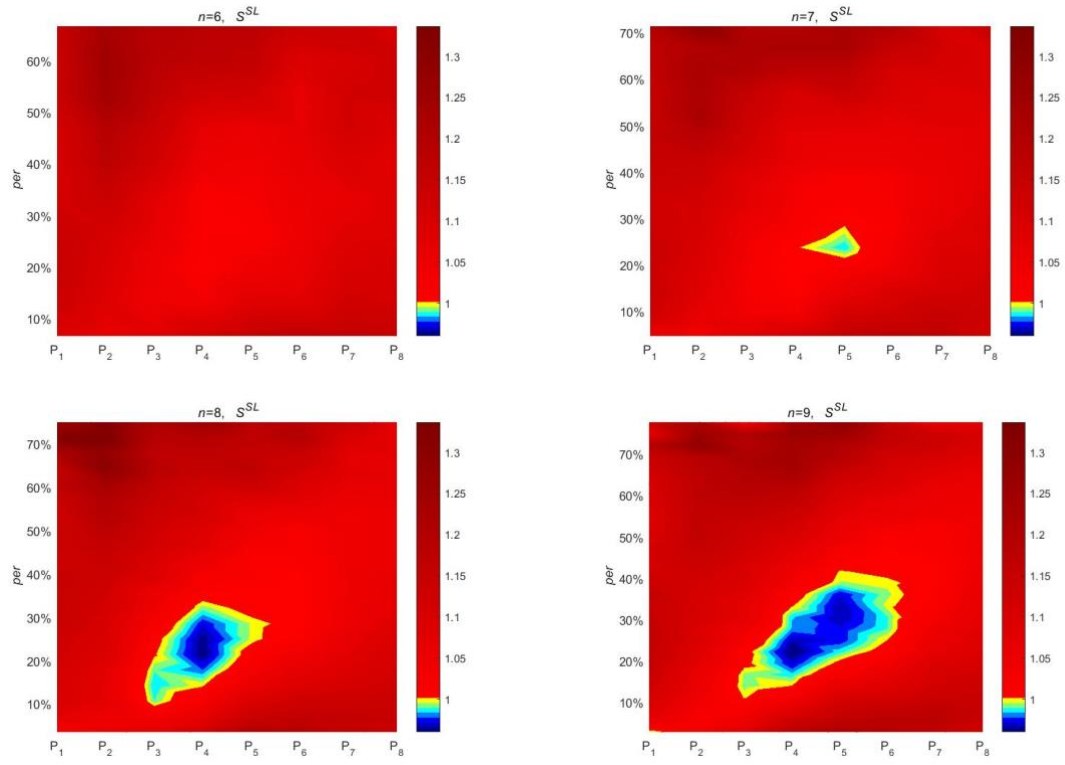
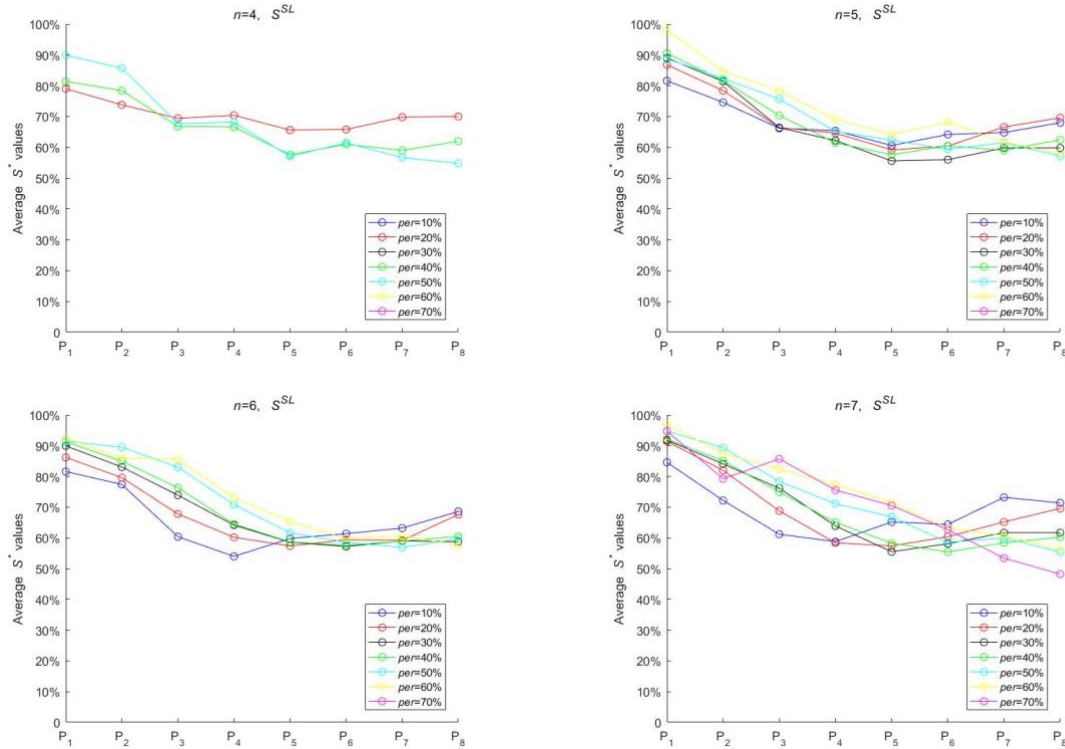


Fig. 3. Average values of R^* under different n , P , and per values.



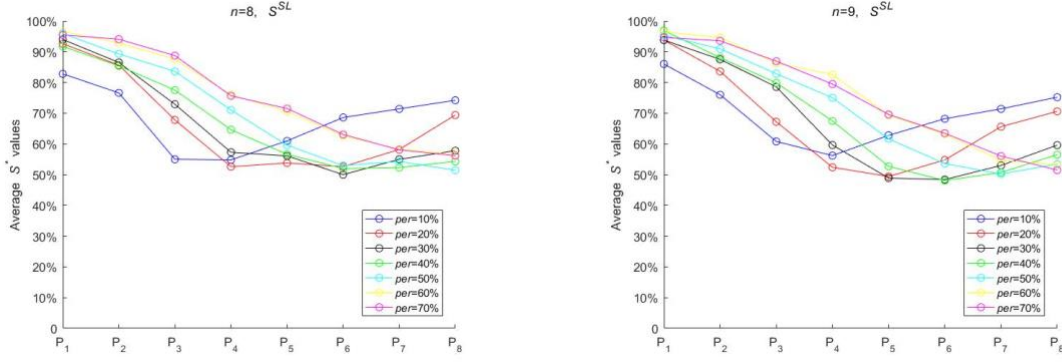


Fig. 4. Average values of S^* under different n , P , and per values.

(1) Fig. 3 reveals how the average values of R^* change with respect to the dimension of the preference relations, the rational degree, and the incompleteness degree; in most cases, the average values of R^* are greater than 1 (the red part in Fig. 3), which means that using self-confident preference relations improve the quality of the priority vectors more than using incomplete preference relations. Furthermore, a small dimension of preference relations (e.g., $n = 4, 5$) clearly enhances the performance of self-confident preference relations.

(2) Fig. 4 further helps us understand the impact of the rational degree and the incompleteness degree of preference relations on the effect of the average values of S^* , which in most cases are higher than 50%, meaning that self-confident preference relations outperform incomplete preference relations. Furthermore, when the rational degree is high (e.g., P_1 and P_2) or low (e.g., P_7 and P_8), the advantage of using self-confident preference relations is more significant than using incomplete preference relations.

Remark 3. Complete preference relations and self-confident preference relations were also compared, and the obtained results show that in most cases self-confident preference relations outperform complete preference relations. This is consistent with the results from Simulation Method I reported in Section 4. However, due to space limitations, this comparative analysis is not included herein.

6. Sensitivity Analysis

In order to make the above conclusions more robust, this section further investigates the sensitivity of the key parameters in the simulation methods.

6.1 Sensitivity of Probability Vectors

Let P_1, P_2, \dots, P_8 be as before. In order to investigate the sensitivity of probability vectors, a random disturbance σ_{ij} generated from a uniform distribution on $[0, 1]$ is added to $P_i = (p_{i1}, p_{i2}, \dots, p_{ij}, \dots, p_{i9})$ ($i = 1, 2, \dots, 8$) (see Eq. (13)), which results in

$P'_i = (p'_{i1}, p'_{i2}, \dots, p'_{ij}, \dots, p'_{i9})$ where

$$p'_{ij} = \frac{p_{ij} + \sigma_{ij}}{\sum_{i=1}^9 (p_{ij} + \sigma_{ij})}. \quad (17)$$

Similarly to the simulation experiments in Section 4.2, Simulation Method I is run 1000 times under different probability vectors $(P'_1, P'_2, \dots, P'_8)$ that are randomly generated by Eq. (17), obtaining average values of R and S as per Figs. 5 and 6, with similar observations as those drawn for Figs. 1 and 2 can be drawn in this case as well. Therefore, it is noticed that conclusions previously discussed in Sections 4 and 5 are not affected by the disturbance of the probability vector used.

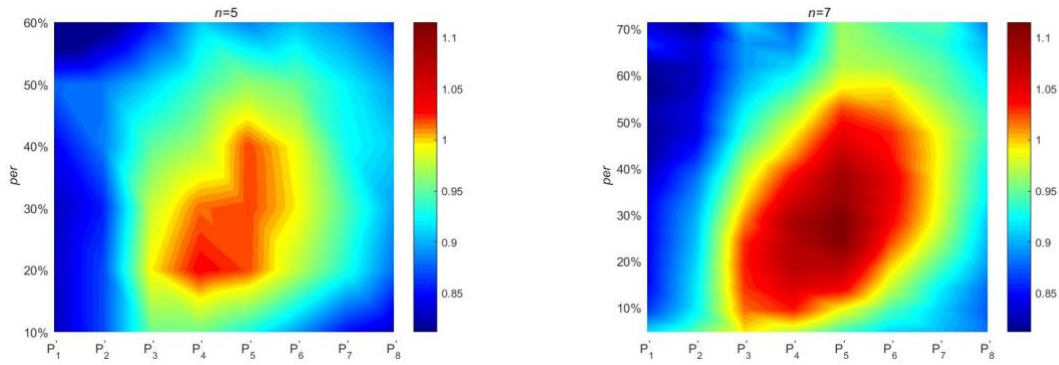


Fig. 5. Average values of R in Simulation Method I under different probability vectors P'_1, P'_2, \dots, P'_8 .

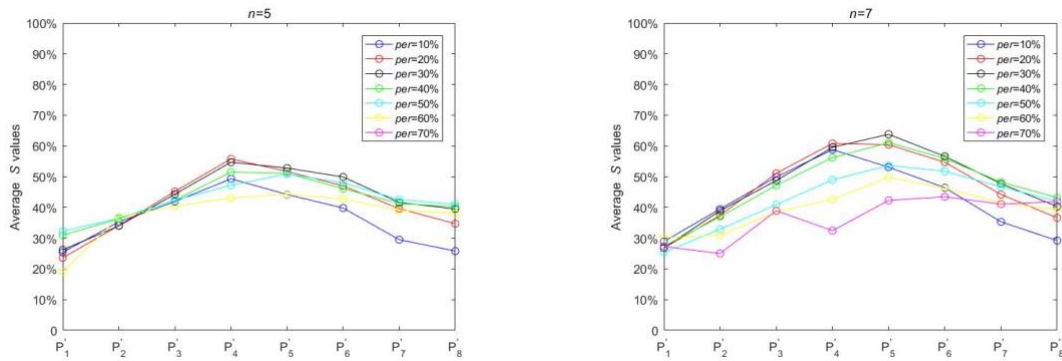


Fig. 6. Average values of S in Simulation Method I under different probability vectors P'_1, P'_2, \dots, P'_8 .

6.2 Sensitivity of Rating Scale

In Simulation Method II, $S^{SL} = \{1, 2, 3, 4, 5\}$ was used as the rating scale to characterize self-confidence levels over preference values. In order to investigate the sensitivity of the rating scale, two alternative rating scales with different granularities, $S^{SL1} = \{1, 2, 3\}$ and

$S^{SL2} = \{1, 2, 3, 4, 5, 6, 7\}$, are also used. The Average values of R^* when running Simulation Method II 1000 times under S^{SL1} and S^{SL2} are shown in Fig. 7. The average values of S^* when Simulation Method II is run 1000 times under S^{SL1} and S^{SL2} are shown in Fig. 8. Again, the same observation drawn from Figs. 3 and 4 can be drawn from Figs. 7 and 8, and self-confident preference relations outperform incomplete preference relations. In particular, it is found that the performance of self-confident preference relations under the rating scale is better than that under the rating scale, and also better than that under the rating scale. This observation suggests that a higher granularity of the rating scale enhances the performance of self-confident preference relations.

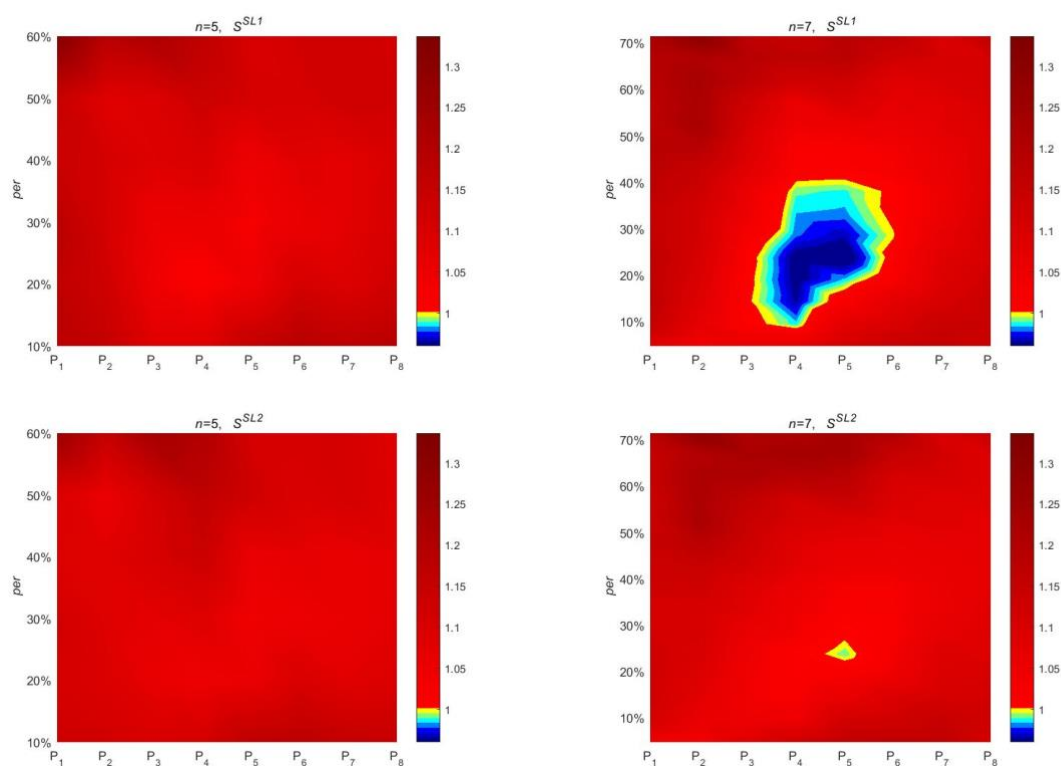
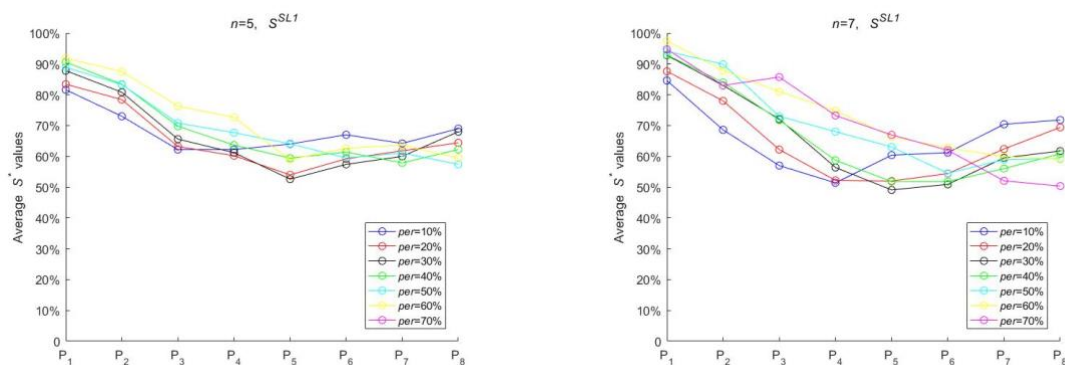


Fig. 7. Average values of R^* in Simulation Method II under S^{SL1} and S^{SL2} .



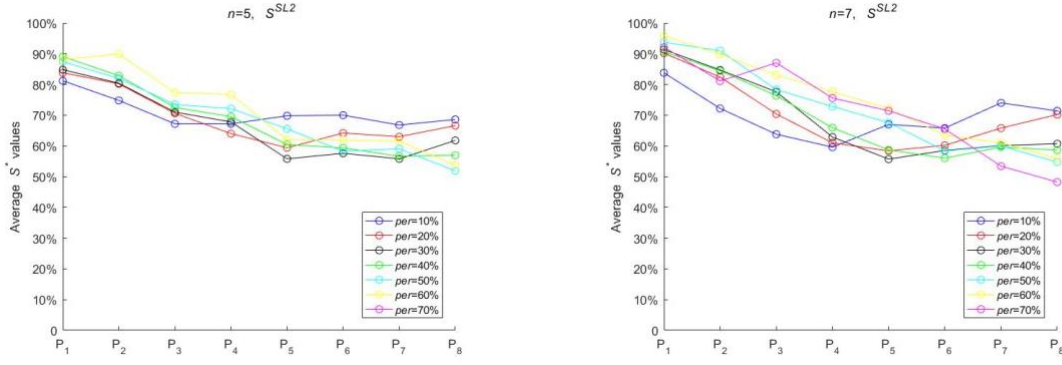


Fig. 8. Average values of S^* in Simulation Method II under S^{SL1} and S^{SL2} .

7. Conclusions

In this paper, we design two bounded rationality principle based simulation experiments to compare the performance of complete preference relations, incomplete preference relations, and self-confident preference relations regarding their accuracy in approximating the true priority vector. The results from the simulation experiments provide evidence towards uncovering answers to the following questions: (1) what are the conditions under which incomplete preference relations outperform complete preference relations? (2) Can self-confident preference relations improve the quality of decisions? The main findings in this research study are:

- (1) The experimental results reveal that incomplete preference relations outperform complete preference relations when the incompleteness degree of preference relations ranges between 20% and 40%, and the decision-maker is of medium rational degree.
- (2) Incomplete preference relations improve the quality of the priority vectors more significantly with the increase of their dimension.
- (3) In most cases, self-confident preference relations outperform incomplete preference relations. Furthermore, a small dimension of preference relations clearly enhances the performance of self-confident preference relations.

Finally, a sensitivity analysis is also reported adding to the robustness of the above findings. In future, we will further compare the performance of complete preference relations, incomplete preference relations, and self-confident preference relations in a group decision-making context [9, 14, 15, 16, 34, 45].

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